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On equivariant homeomorphisms of boundaries of CAT(0) groups and Coxeter groups

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1. Introduction

In this note, we introduce on equivariant homeomorphisms of boundaries of CAT(0) groups (and Coxeter groups) and (boundary-)rigidity in [17].

A *geometric* action on a CAT(0) space is an action by isometries which is proper and cocompact. We note that every CAT(0) space X on which some group G acts geometrically is a proper space and we can consider its ideal boundary ∂X (cf. [4], [11]). A group G is called a *CAT(0) group*, if G acts geometrically on some CAT(0) space X.

It is well-known that if a Gromov hyperbolic group G acts geometrically on two negatively curved spaces X and Y, then the natural quasi-isometry $\phi : Gx_0 \to Gy_0$ $(gx_0 \mapsto gy_0)$ extends continuously to a G-equivariant homeomorphism $\bar{\phi}$: $\partial X \to \partial Y$ of the boundaries of X and Y (cf. [4], [5], [11], [12], [13]).

M. Gromov [13] asked whether the boundaries of two CAT(0) spaces X and Y are G-equivariant homeomorphic whenever a CAT(0) group G acts geometrically on the two CAT(0) spaces X and Y. P. L. Bowers and K. Ruane [3] have constructed an example that the natural quasi-isometry $Gx_0 \to Gy_0$ $(gx_0 \mapsto gy_0)$ does not extend continuously to any map between the boundaries ∂X and ∂Y of X and Y. Also, C. Croke and B. Kleiner [6] have constructed a CAT(0) group G which acts geometrically on two CAT(0) spaces X and Y whose boundaries are not homeomorphic, and J. Wilson [26] has proved that this CAT(0) group has uncountably many boundaries.

In this note, we suppose that a CAT(0) group G acts geometrically on two CAT(0) spaces X and Y. Let $x_0 \in X$ and $y_0 \in Y$.

Then we consider the following question.

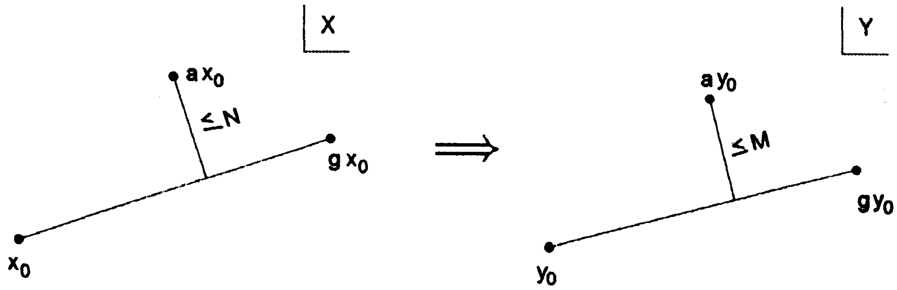
Question. When does the quasi-isometry $\phi : Gx_0 \to Gy_0$ $(gx_0 \mapsto gy_0)$ continuously extend to a G-equivariant homeomorphism $\bar{\phi} : \partial X \to \partial Y$ of the boundaries?

$$\begin{array}{cccccc} & X & \supset & Gx_0 & \longleftrightarrow & \partial X \\ G & & & \phi\downarrow & & \downarrow\bar{\phi}\,? \\ & Y & \supset & Gy_0 & \longleftrightarrow & \partial Y \end{array}$$

2. Main theorems

The following condition $(*)$ comes from observating the Bowers-Ruane's example.

$(*)$ There exist constants $N > 0$ and $M > 0$ such that $GB(x_0, N) = X$, $GB(y_0, M) = Y$ and for any $g, a \in G$, if $[x_0, gx_0] \cap B(ax_0, N) \neq \emptyset$ in X then $[y_0, gy_0] \cap B(ay_0, M) \neq \emptyset$ in Y.



Then we obtain the following theorem.

Theorem 1 ([17]). *If the condition $(*)$ holds, then the quasi-isometry $\phi : Gx_0 \to Gy_0$ $(gx_0 \mapsto gy_0)$ continuously extends to a G-equivariant homeomorphism $\bar{\phi} : \partial X \to \partial Y$ of the boundaries.*

We also consider the following condition $(**)$.

$(**)$ For any sequence $\{g_i \mid i \in \mathbb{N}\} \subset G$, the sequence $\{g_i x_0 \mid i \in \mathbb{N}\}$ is a Cauchy sequence in $X \cup \partial X$ if and only if the sequence $\{g_i y_0 \mid i \in \mathbb{N}\}$ is a Cauchy sequence in $Y \cup \partial Y$.

Then we also obtain the following theorem.

Theorem 2 ([17]). *The condition $(**)$ holds if and only if the quasi-isometry $\phi : Gx_0 \to Gy_0$ $(gx_0 \mapsto gy_0)$ continuously extends to a G-equivariant homeomorphism $\bar{\phi} : \partial X \to \partial Y$ of the boundaries.*

3. Rigidity of boundaries

In this note, a CAT(0) group G is said to be *(boundary-)rigid*, if G determines its ideal boundary up to homeomorphisms, i.e., all boundaries of CAT(0) spaces on which G acts geometrically are homeomorphic.

Also a CAT(0) group G is said to be *equivariant (boundary) rigid*, if G determines its ideal boundary by the equivariant homeomorphisms as above (i.e., if for any two CAT(0) spaces X and Y on which G acts geometrically the quasi-isometry $\phi : Gx_0 \to Gy_0$ $(gx_0 \mapsto gy_0)$ continuously extends to a G-equivariant homeomorphism $\bar{\phi} : \partial X \to \partial Y$ of the boundaries).

As an application of Theorem 1, we can obtain examples of equivariant rigid CAT(0) groups.

Example ([17]). Any group of the form

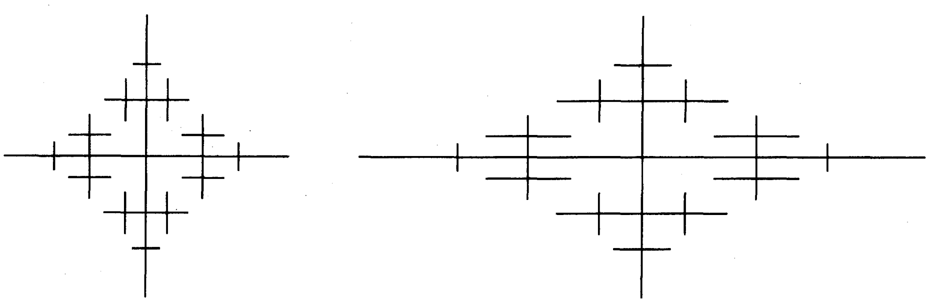
$$\mathbb{Z}^{n_1} * \cdots * \mathbb{Z}^{n_k} * A_1 * \cdots * A_l$$

where $n_i \in \mathbb{N}$ and each A_j is a finite group is an equivariant rigid CAT(0) group.

As an application of Theorem 2, we can also obtain examples of non equivariant rigid CAT(0) groups.

Example ([17]). Let $G = F_2 \times \mathbb{Z}$, where F_2 is the rank 2 free group generated by $\{a, b\}$. Let T and T' be the Cayley graphs of F_2 with respect to the generating set $\{a, b\}$ such that

(1) in T, all edges $[g, ga]$ and $[g, gb]$ $(g \in F_2)$ have the unit length, and
(2) in T', the length of $[g, ga]$ is 2 and the length of $[g, gb]$ is 1 for any $g \in F_2$.

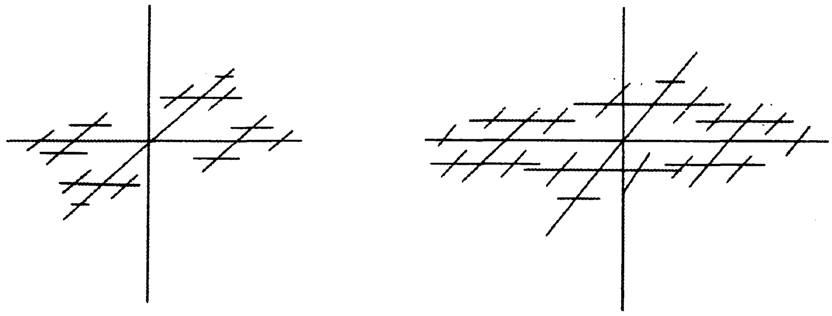


T $\qquad\qquad$ T'

Here we note that F_2 acts naturally and geometrically on T and T'.

Let $X = T \times \mathbb{R}$ and $Y = T' \times \mathbb{R}$.

We consider the natural actions of the group $G = F_2 \times \mathbb{Z}$ on the CAT(0) spaces X and Y. Then the group G acts geometrically on the two CAT(0) spaces X and



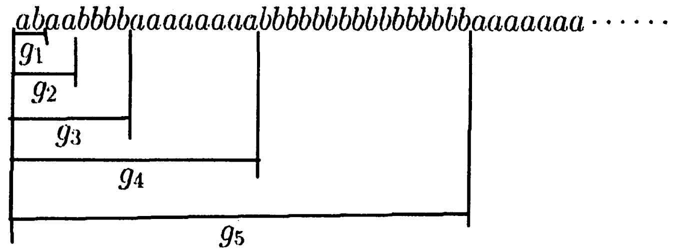
$X = T \times \mathbb{R}$ $\qquad\qquad\qquad$ $Y = T' \times \mathbb{R}$

Y, and the quasi-isometry $gx_0 \mapsto gy_0$ (where $x_0 = (1,0) \in X$ and $y_0 = (1,0) \in Y$) does not extend continuously to any map from ∂X to ∂Y.

Indeed, we can consider the sequence $\{g_n \mid n \in \mathbb{N}\} \subset F_2$ such that $g_1 = ab$ and

$$g_n = \begin{cases} g_{n-1}a^{2^{n-1}} & \text{if } n \text{ is even} \\ g_{n-1}b^{2^{n-1}} & \text{if } n \text{ is odd} \end{cases}$$

for $n \geq 2$. Here we note that the length of the words of g_n in F_2 is 2^n.



Let $\bar{g}_n = (g_n, 2^n) \in F_2 \times \mathbb{Z}$ for $n \in \mathbb{N}$. Then $\{\bar{g}_n x_0\}$ is a Cauchy sequence in $X \cup \partial X$. On the other hand, $\{\bar{g}_n y_0\}$ is *not* a Cauchy sequence in $Y \cup \partial Y$ (see Figure 1).

Hence, the quasi-isometry $\phi : Gx_0 \to Gy_0$ $(gx_0 \mapsto gy_0)$ does not continuously extend to any map $\bar{\phi} : \partial X \to \partial Y$ of the boundaries.

Remark ([17]).

- $G = F_2 \times \mathbb{Z}$ is a non equivariant rigid CAT(0) group.
- $G = F_2 \times \mathbb{Z}$ is a rigid CAT(0) group whose boundary is the suspension of the Cantor set.

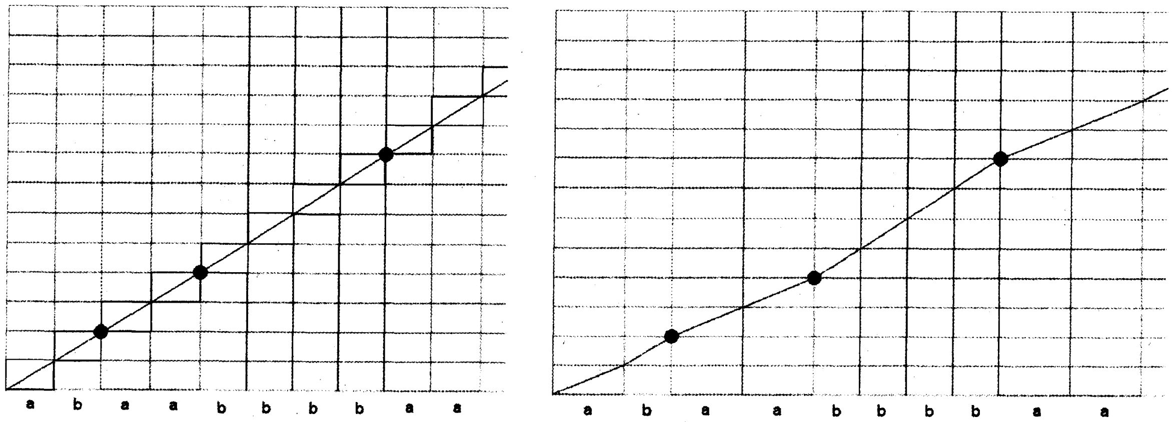


FIGURE 1

- By the same idea, every CAT(0) group of the form $G = F \times H$ where F is a free group of rank $n \geq 2$ and H is an infinite CAT(0) group, is non equivariant rigid.

4. COXETER GROUPS ACTING CAT(0) SPACES AS REFLECTION GROUPS

A Coxeter group W is said to be *equivariant rigid as a reflection group*, if for any two CAT(0) spaces X and Y on which W acts geometrically as reflection groups, the quasi-isometry $\phi : Wx_0 \to Wy_0$ ($wx_0 \mapsto wy_0$) where $x_0 \in X$ and $y_0 \in Y$ continuously extends to a W-equivariant homeomorphism $\bar{\phi} : \partial X \to \partial Y$ of the boundaries.

Theorem 3 ([17]). *The following statements hold.*

(i) *If Coxeter groups W_1 and W_2 are equivariant rigid as reflection groups, then so is $W_1 * W_2$.*

(ii) *For a Coxeter group $W = W_A *_{W_{A\cap B}} W_B$ where $W_{A\cap B}$ is finite, if W determines its Coxeter system up to isomorphism, and if W_A and W_B are equivariant rigid as reflection groups then so is W, where W_T is the parabolic subgroup of W generated by T.*

Corollary 4 ([17]). *Any group of the form*

$$W = W_1 * \cdots * W_n$$

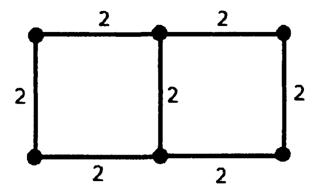
where each W_i is a Gromov hyperbolic Coxeter group, an affine Coxeter group or a finite Coxeter group, is an equivariant rigid as a reflection group.

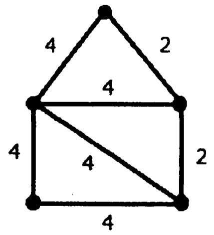
Corollary 5 ([17]). *Any Coxeter group of the form*

$$W = (\cdots(W_{A_1} *_{W_{B_1}} W_{A_2}) *_{W_{B_2}} W_{A_3}) * \cdots) *_{W_{B_{n-1}}} W_{A_n}$$

where each W_{A_i} is a Gromov hyperbolic Coxeter group, an affine Coxeter group or a finite Coxeter group, each W_{B_i} is finite and W determines its Coxeter system up to isomorphism, is an equivariant rigid as a reflection group.

Example. The Coxeter groups defined by the following diagrams are equivariant rigid as reflection groups.





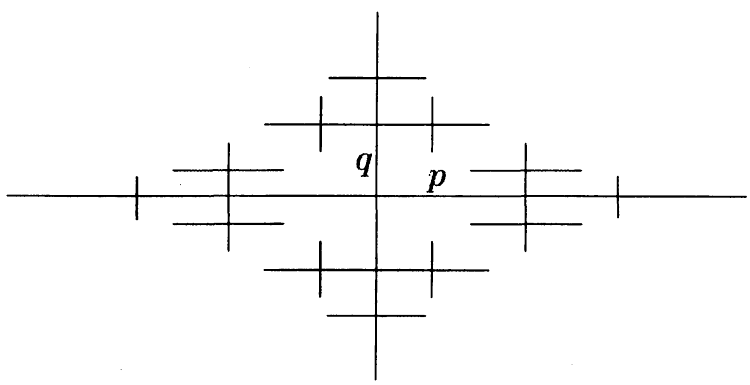
5. Conjecture

Now we introduce a conjecture.

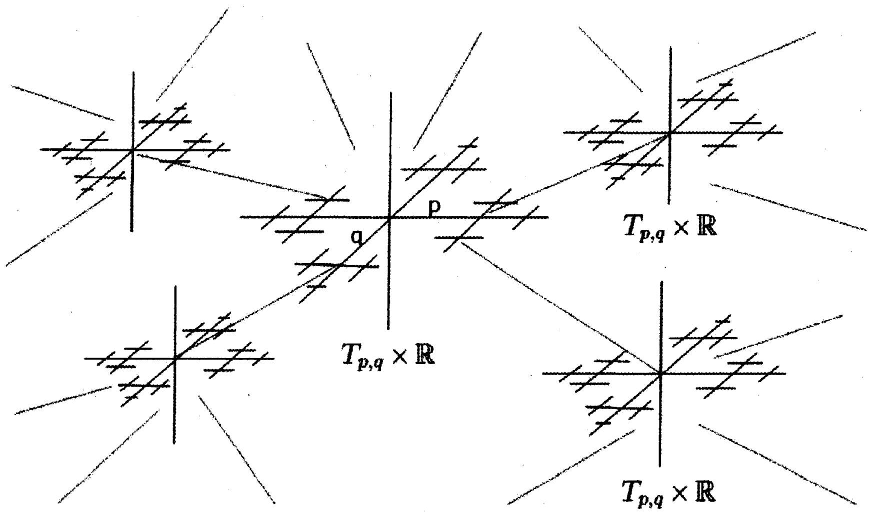
Conjecture ([17]). The group $G = (F_2 \times \mathbb{Z}) * \mathbb{Z}_2$ will be a non-rigid CAT(0) group with uncountably many boundaries.

For $p \geq q \geq 1$, let $T_{p,q}$ be the Cayley graph of the free group F_2 with the generating set $\{a, b\}$ such that

- the length of $[g, ga]$ is p and the length of $[g, gb]$ is q for any $g \in F$.



$T_{p,q}$



$\Sigma_{p,q}$

Then $F_2 \times \mathbb{Z}$ acts naturally on $T_{p,q} \times \mathbb{R}$. We can construct a *cuboidal* cell complex $\Sigma_{p,q}$ on which $G = (F_2 \times \mathbb{Z}) * \mathbb{Z}_2$ acts geometrically, where the 1-skeleton of $\Sigma_{p,q}$ is the Cayley graph of G and $T_{p,q} \subset \Sigma^{(1)}_{p,q}$.

Then, the author thinks that if $\frac{p}{q} \neq \frac{p'}{q'}$ then the boundaries $\partial\Sigma_{p,q}$ and $\partial\Sigma_{p',q'}$ will be not homeomorphic.

6. On rigidity

Finally, we introduce problems of rigidity in group actions.

Let G and H be groups acting geometrically (i.e. properly and cocompactly by isometries) on metric spaces (X, d_X) and (Y, d_Y) respectively. We consider orbits $Gx_0 \subset X$ and $Hy_0 \subset Y$ where $x_0 \in X$ and $y_0 \in Y$.

Let $\phi : G \to H$ be a map and let $\phi' : Gx_0 \to Hy_0$ $(gx_0 \mapsto \phi(g)y_0)$.

Here if X and Y are Gromov hyperbolic spaces, CAT(0) spaces or Busemann spaces, then we can define the boundaries ∂X and ∂Y.

Then it is well-known that if $\phi : G \to H$ is an isomorphism then $\phi' : Gx_0 \to Hy_0$ is a quasi-isometry and moreover if G is Gromov hyperbolic then ϕ' induces an equivariant homeomorphism $\bar{\phi} : \partial X \to \partial Y$.

Theorem 2 implies that if $\phi : G \to H$ is an isomorphism and the map $\phi' : Gx_0 \to Hy_0$ satisfies the condition $(**)$ then ϕ' induces an equivariant homeomorphism $\bar{\phi} : \partial X \to \partial Y$.

$$\begin{array}{ccccccc} G & \stackrel{\cdot}{\curvearrowright} & X & \supset & Gx_0 & \longleftrightarrow & \partial X \\ \downarrow\phi & & & & \downarrow\phi' & & \downarrow\bar{\phi} \\ H & \stackrel{\cdot}{\curvearrowright} & Y & \supset & Hy_0 & \longleftrightarrow & \partial Y \end{array}$$

Then there are problems of rigidity.

(I) If $\phi : G \to H$ is an isomorphism then when does there exist an homeomorphism $\bar{\phi} : \partial X \to \partial Y$?

(II) If $\phi : G \to H$ is an isomorphism then when does ϕ' induce an equivariant homeomorphism $\bar{\phi} : \partial X \to \partial Y$?

(III) If $X = Y$ and $Gx_0 = Hx_0$ then when are groups G and H virtually isomorphic (i.e. there exist finite-index subgroups G' and H' of G and H respectively such that G' and H' are isomorphic)?

(IV) If $X = Y$ and $Gx_0 = Hx_0$ then when do there exist finite-index subgroups G' and H' of G and H respectively such that G' and H' are conjugate in the isometry group $\mathrm{Isom}(X)$ of X?

(V) If there is an isomorphism $\phi : G \to H$ then when does there exist a homeomorphism (or homotopy equivalence) $\psi : X/G \to Y/H$?

Here it seems that (III)–(V) are relate to [1], [8], [9], [14], [18], [19], [20], [22] and [23].

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